4 INTEGRATION

Objectives

In this chapter you will learn about:

- Identify definite and indefinite integrals.
- Integrate polynomials using power rule.
- Apply standard forms of integrals as converse of differentiation.
- Calculate the area between a curve and the x-axis or by a curve, the x-axis and the ordinates.

Introduction

In Mathematics many processes can be reversed. For example, the reverse of addition is subtraction, the reverse process of multiplication is division, etc. These are not the only mathematical processes that can be reversed. In the previous chapter, you learned the process of differentiation. Like addition, multiplication, division, etc. the process of differentiation can be reversed. The reverse process of differentiation is called integration. In this chapter we will provide some integration rules to help you to find anti-derivative.

The derivative of $f(x)=x^2$ is $f'(x)=2x$. How do we reverse $f'(x)=2x$ to get $f(x)$?

We are going to use the mathematical process called integration.

4.1 The Indefinite integrals

Suppose we differentiate the function $f(x)=x^2$. We get $f'(x)=2x^{2-1}=2x$.

Integration reverses this process and using mathematical language we say that the integral of $2x$ is x^2.

We can represent the relationship between integration and differentiation as follows:

$$\begin{tikzcd} x^2 \arrow[r, bend left, "\text{Differentiate}"] & 2x \arrow[l, bend left, "\text{Integrate}"] \end{tikzcd}$$

If we differentiate the following functions:

$f(x)=x^2+100$ $\qquad$ $f(x)=x^2-7$ $\qquad$ $f(x)=x^2+0,75$

we get $2x$.

Therefore, we notice that the above functions give $2x$ when differentiated.

As you can see the above functions have a constant term which becomes zero when we differentiate the function. When we integrate a function we have no way of telling what the original constant term might have been. All we can do is to acknowledge the existence of such a constant term by including in our answer an unknown constant, C, called the **constant of integration. We now say that the integral of $2x$ is x^2+C.**

We write the statement "the integral of $2x$ is x^2+C" mathematically as follows: $\int 2x\,dx = x^2+C$

We use the symbol $\int$ to show that we are integrating, *dx* tells us that the function we are integrating is written in terms of *x*. The function being integrated ($2x$) is called the **integrand**.

$x^2 + C$ is called the **indefinite integral** of $2x$ with respect to x. **The indefinite integral represents many possible anti-derivatives.**

To integrate a power of x increase the power by 1and divide the answer you get by the new power. Mathematically we write this statement as follows:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

Going back to our example we have:

$$\int 2x \, dx = \frac{2x^{1+1}}{1+1} + C$$

$$= \frac{2x^2}{2} + C$$

$$= x^2 + C$$

The integral of $kx^n, n \neq -1$

A constant factor in an integral can be moved outside the integral sign

$$\int kx^n \, dx = k\int x^n \, dx$$

Worked Example

(a) $\int 3 \, dx$

$$\int 3 \, dx = \int 3x^0 \, dx$$

$$= 3\int x^0 \, dx$$

$$= 3\left(\frac{x^{0+1}}{0+1}\right) + C$$

$$= 3\left(\frac{x}{1}\right) + C$$

$$= 3x + C$$

Worked Example

Find the following integrals:

(b) $\int 3x^3 dx$

$$\int 3x^3 dx = 3\int x^3 dx$$

$$3\int x^3 dt = 3\left(\frac{x^{3+1}}{3+1}\right)+C$$

$$=3\left(\frac{x^4}{4}\right)+C$$

$$=\frac{3x^4}{4}+C$$

Exercise 1

Find the following indefinite integrals

1.1 $\int 5\ dx$

1.2 $\int (-3)dx$

1.3 $\int x^3 dx$

1.4 $\int 6x^5\ dx$

1.5 $\int 8x^7\ dx$

1.6 $\int -x\ dx$

1.7 $\int x^4\ dx$

1.8 $-\int 3x\, dx$

1.9 $\int \frac{1}{2}x\ dx$

The integral of $f(x)+g(x)$ **or of** $f(x)-g(x)$

We can integrate the sum or difference of two or more functions by integrating each term separately.

Worked example

(b) $\int\left(4x^3-2x^2+x-2\right)dx$

$$\int\left(4x^3-2x^2+x-2\right)dx=\int 4x^3\ dx-\int 2x^2\ dx+\int x\ dx-\int 2\ dx$$

$$=4\int x^3\ dx-2\int x^2\ dx+\int x\ dx-2\int x^0\ dx$$

$$=4\left(\frac{x^{3+1}}{3+1}\right)-2\left(\frac{x^{2+1}}{2+1}\right)+\frac{x^{1+1}}{1+1}-2\left(\frac{x^{0+1}}{0+1}\right)+C$$

$$=4\left(\frac{x^4}{4}\right)-2\left(\frac{x^3}{3}\right)+\frac{x^2}{2}-2\left(\frac{x}{1}\right)+C$$

$$=x^4-\frac{2x^3}{3}+\frac{x^2}{2}-2x+C$$

Exercise 2

Find each integral.

2.1 $\int\left(x+1\right)dx$

2.2 $\int\left(-x+1\right)dx$

2.3 $\int(3x+2)dx$

2.4 $\int\left(1-x^2\right)dx$

2.5 $\int\left(1-x^3\right)dx$

2.6 $\int\left(x^2-2x+1\right)dx$

2.7 $\int\left(x^3-2x^2+x-5\right)dx$

2.8 $\int(3x^4-x^5)dx$

2.9 $\int\left(5+3x+x^2\right)dx$

2.10 $\int\left(-3x^2+\frac{1}{2}x+1\right)dx$

The integral of the form $\int\frac{1}{x}dx,\ x\neq 0$

To calculate integrals of the form $\int\frac{1}{x}dx$ we use the general logarithm rule $\int\frac{1}{x}dx=\ln|x|+C, x\neq 0$.

Worked Example

Find the following integrals

$\int\frac{5}{x}dx$

$$\int\frac{5}{x}\ dx=5\int\frac{1}{x}\ dx$$

$$=5\ln|x|+C$$

Worked Example

$\int \frac{6}{5x}\, dx$

$\int \frac{6}{5x}\, dx = \int \frac{6}{5}\left(\frac{1}{x}\right) dx$

$= \frac{6}{5}\int \frac{1}{x}\, dx$

$= \frac{6}{5}\ln|x| + C$

Exercise 3

Find the following integrals

3.1 $\int \frac{7}{x} dx$ 3.2 $\int \frac{100}{x}\, dx$ 3.3 $\int \frac{23}{x}\, dx$

3.4 $\int \frac{\sqrt{3}}{x}\, dx$ 3.5 $\int \frac{\sqrt{5}}{x}\, dx$ 3.6 $\int \frac{\sqrt{11}}{x}\, dx$

3.7 $\int \frac{2}{3x}\, dx$ 3.8 $\int \frac{2}{5x}\, dx$ 3.9 $\int \frac{-1}{3x}\, dx$

4.2 Definite integrals

The quantity $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ from a to b. The numbers a and b are called the limits of integration. They tell us the interval we are working in. The number a is called the lower limit and b is called the upper limit.

If $f(x)$ is the integral of $f(x)$, then $\int_a^b f(x)\,dx = f(b) - f(a)$. To evaluate the definite integral we:

- integrate the given function but we do not include the constant of integration
- then substitute the value of the upper limit into the integral $f(b)$
- substitute the value of the lower limit into the integral $f(a)$
- subtract the value of the lower from that of the upper limit.
  i.e we calculate $f(b) - f(a)$
- The answer is a number

Worked Examples

Evaluate the following:

(a) $\int_1^2 2x\ dx$

(b) $\int_1^3 \left(3x - x^3\right) dx$

Solutions

(a) $\int_1^2 2x\ dx$

$$\int_1^2 2x\ dx = 2\int_1^2 x\ dx$$

$$= \left[\frac{2x^{1+1}}{1+1}\right]_1^2$$

$$= \left[\frac{2x^2}{2}\right]_1^2$$

$$= \left[x^2\right]_1^2$$

$$= (2)^2 - (1)^2$$

$$= 4 - 1$$

$$= 3$$

(b) $\int_1^3 \left(3x - x^3\right) dx$

$$f(x) = \int \left(3x - x^3\right) dx$$

$$= \int 3x\ dx - \int x^3 dx$$

$$= \left[\frac{3x^{1+1}}{1+1}\right] - \left[\frac{x^{3+1}}{3+1}\right]$$

$$= \frac{3x^2}{2} + \frac{x^4}{4}$$

$$\int_1^3 \left(3x - x^3\right) dx = f(3) - f(1)$$

$$f(3) = \frac{3(3)^2}{2} + \frac{(3)^4}{4}$$

$$= \frac{27}{2} + \frac{81}{4}$$

$$= \frac{135}{4}$$

$$f(1) = \frac{3(1)}{2} + \frac{(1)^4}{4}$$

$$= \frac{3}{2} + \frac{1}{4}$$

$$= \frac{7}{4}$$

$$f(3) - f(1) = \frac{135}{4} - \frac{7}{4}$$

$$= \frac{128}{4}$$

$$= 32$$

The constant C is omitted when using the integration formulas to find a definite integral.

Exercise 4

Calculate:

4.1 $\int_0^2 \left(x^2 + x\right) dx$

4.2 $\int_1^3 \left(3u^3 - 1\right) du$

4.3 $\int_{-2}^{-1} \left(x^3 - 2x\right) dx$

4.4 $\int_{-2}^{1} \left(6x^2 - 5x + 2\right) dx$

4.5 $\int_0^4 \sqrt{t}\left(t + 2\right) dt$

4.6 $\int_0^1 \left(4x - 6x^{\frac{2}{3}}\right) dx$

4.7 $\int_{-2}^{0} \left(\frac{1}{3}x^2 - 2\right) dx$

4.8 $\int_2^3 \left(\frac{1}{2}x^4 + x\right) dx$

Using integration to find the area included by a curve and the *x*-axis or by a curve the *x*-axis and the ordinates $x - a$ and $x - b$, where $a, b \in Z$.

Calculating the shaded area below the curve as shown in the picture below can be very difficult. The shaded shape is not a polygon. The upper part is a curve not a straight line. We can try and divide the shaded area into rectangles and find the sum of the areas of the rectangles. The trick is to make the rectangles as thin as possible and to use as many rectangles as possible. This is a tedious exercise.

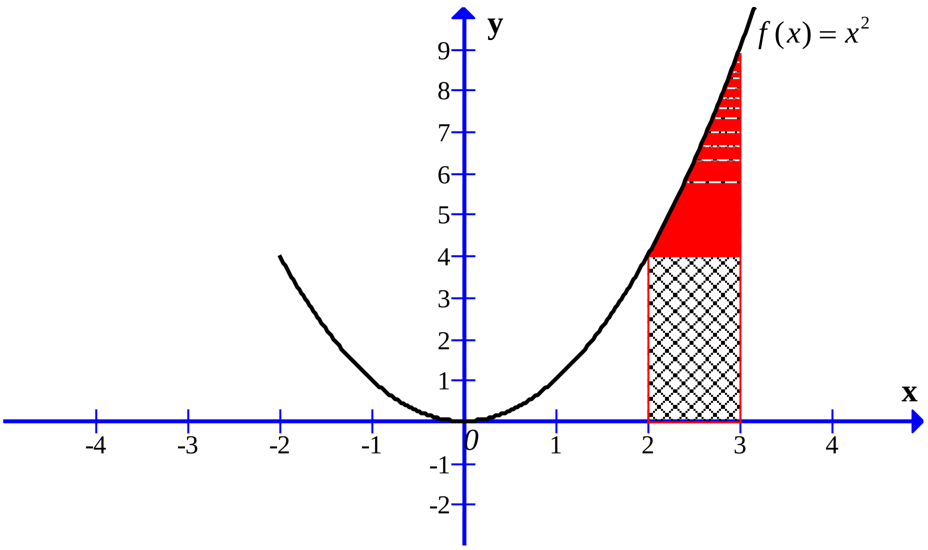
Thanks to the mathematicians who proved that these areas can be found by the reverse process of differentiating which is integration. We do not have to calculate thousands of rectangle areas in order to find the area of shapes like the one below.

We use the definite integral to find the area between the curve and the *x*-axis on the given interval. The definite integral $\int_a^b f(x)dx$ will give us the area under the curve of f between *a* and *b*. The area of the shaded region below is defined by $\int_2^3 (x^2)dx$.

Worked examples

Calculate the shaded area using integration.

a)

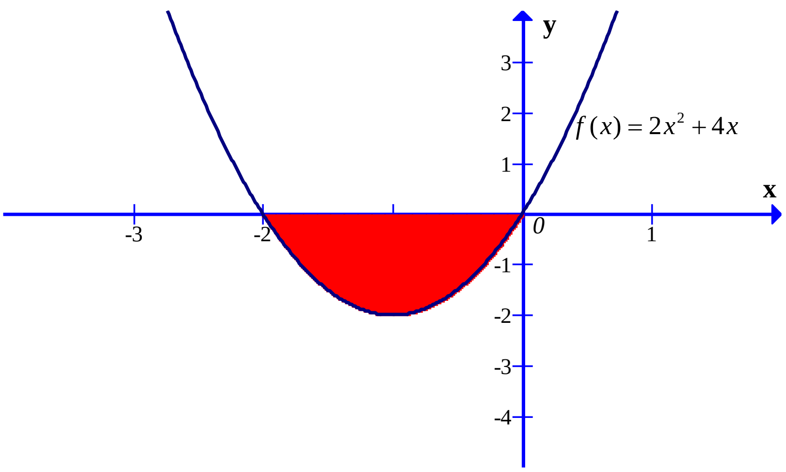


The shaded area = $\int_2^3 (x^2)dx$

$$= \left[\frac{x^3}{3}\right]_2^3 = \frac{3^3}{3} - \frac{2^3}{3} = \frac{19}{3}$$ square units.

The definite integral will produce a positive value if $f(x) > 0$ in the same interval.

b)



$$\left[\frac{2x^3}{3}+2x^2\right]_{-2}^{0}=\left[\frac{2(0)^3}{3}+2(0)^2\right]-\left[\frac{2(-2)^3}{3}+2(-2)^2\right]$$

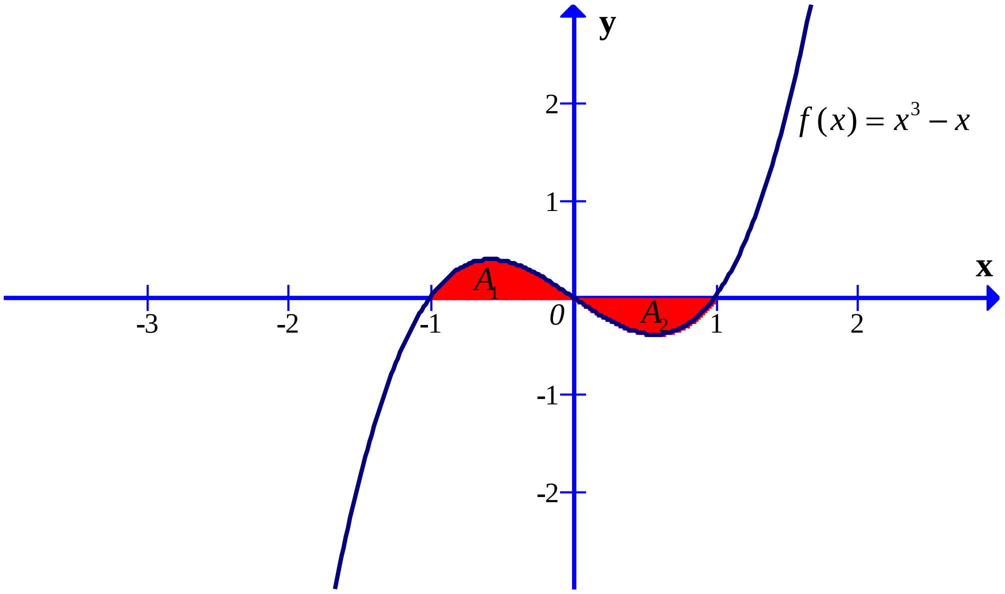
$$=0-\left[\frac{-16}{3}+8\right]$$

$$=\left[\frac{16}{3}-8\right]$$

$$=\left[\frac{16-24}{3}\right]=\left[\frac{-8}{3}\right]=\frac{8}{3}\text{ square units}$$

The definite integral produce a negative value if $f(x) < 0$ in the interval $x = -2$ to $x = 0$. Since we are calculating the area, and the area is always a positive number, the absolute value is used. The absolute value of a number is always positive.

Find the total area between $f(x) = x^3 - x$ and the x-axis from x = -1 to x = 1.



$$A_1 = \int_{-1}^{0} (x^3 - x)dx$$

$$\left[\frac{x^4-2x^2}{4}\right]_{-1}^{0}=\frac{(0)^4-2(0)^2}{4}-\frac{(-1)^4-2(-1)^2}{4}$$

$$=\left[\frac{1}{4}\right]$$

$$A_2 = \int_{0}^{1} (x^3 - x)dx$$

$$\left[\frac{x^4-2x^2}{4}\right]_{-1}^{0}=\frac{(1)^4-2(1)^2}{4}-\frac{(0)^4-2(0)^2}{4}$$

$$=\left[-\frac{1}{4}\right]$$

$$A = A_1 + A_2$$

$$=\left[\frac{1}{4}\right]+\left[-\frac{1}{4}\right]$$

The total area is $\frac{1}{2}$ square unit.

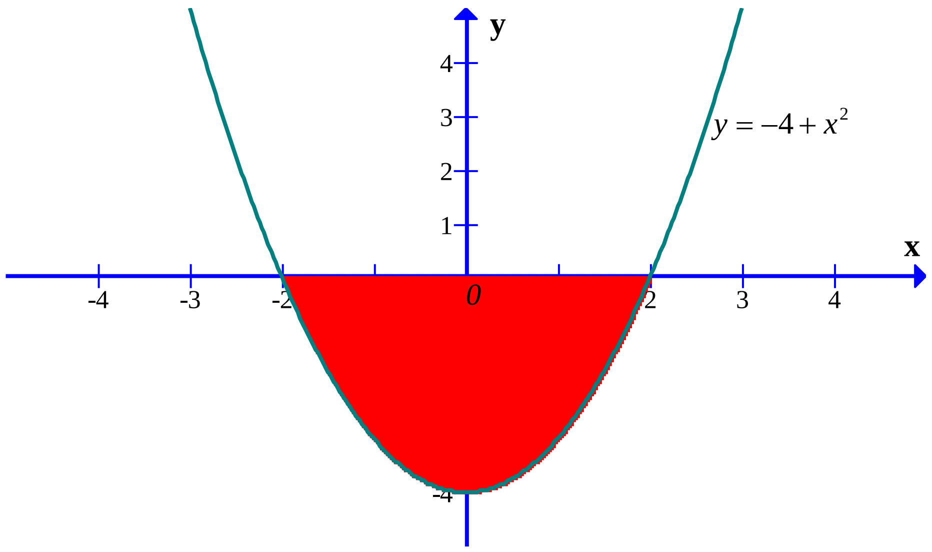
NOTE

When a function $f(x)$ is both positive and negative in the given interval like the example above, and the area is to be found, the limits of integration must be split at the zeros of the function. Areas A_1 and A_2 were calculated separately. The total area is the sum of two areas.

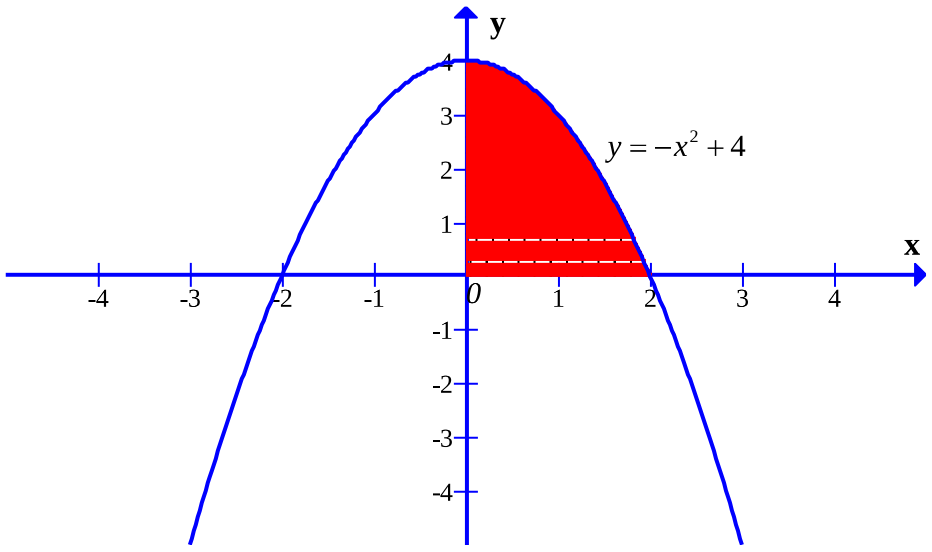
Exercise 5

Use integration to find the shaded areas of the following.

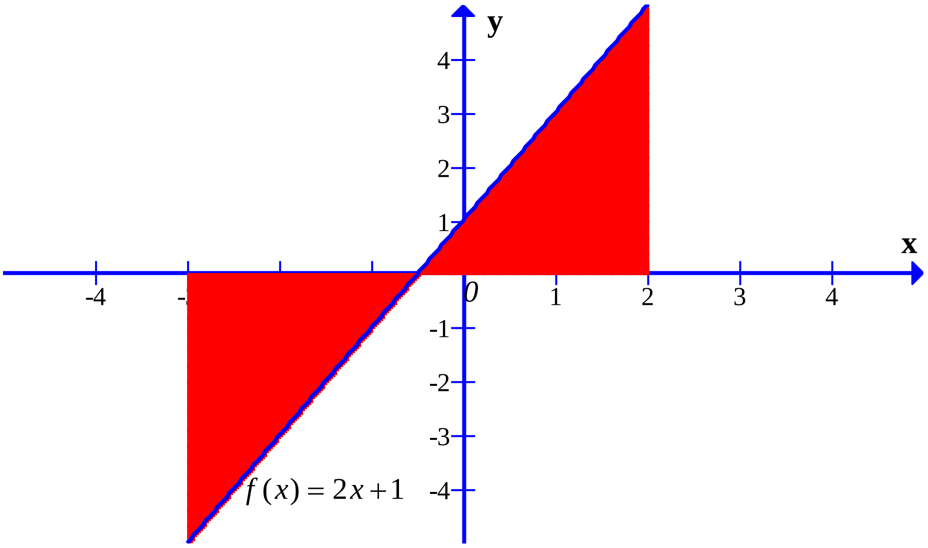
5.1



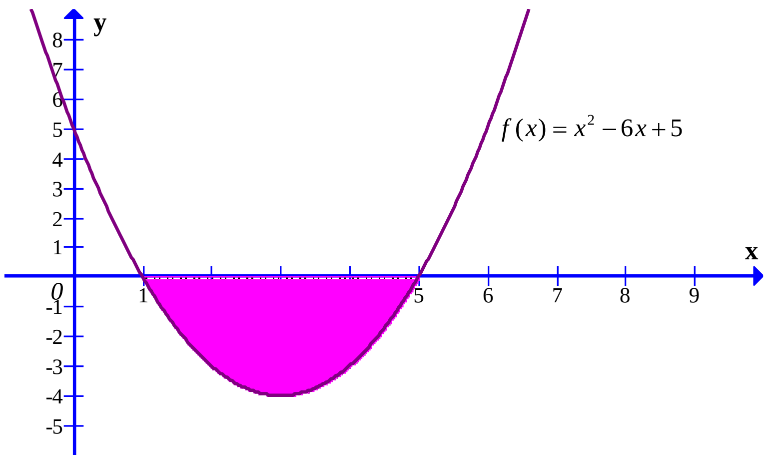
5.2



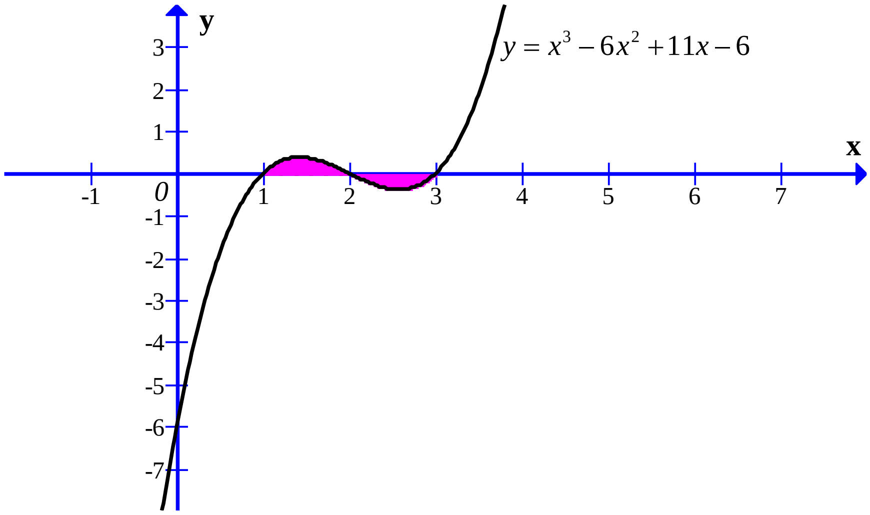
5.3


y
x
4
3
2
1
-4
0
1
2
3
4
-1
-2
-3
-4
$f(x) = 2x + 1$

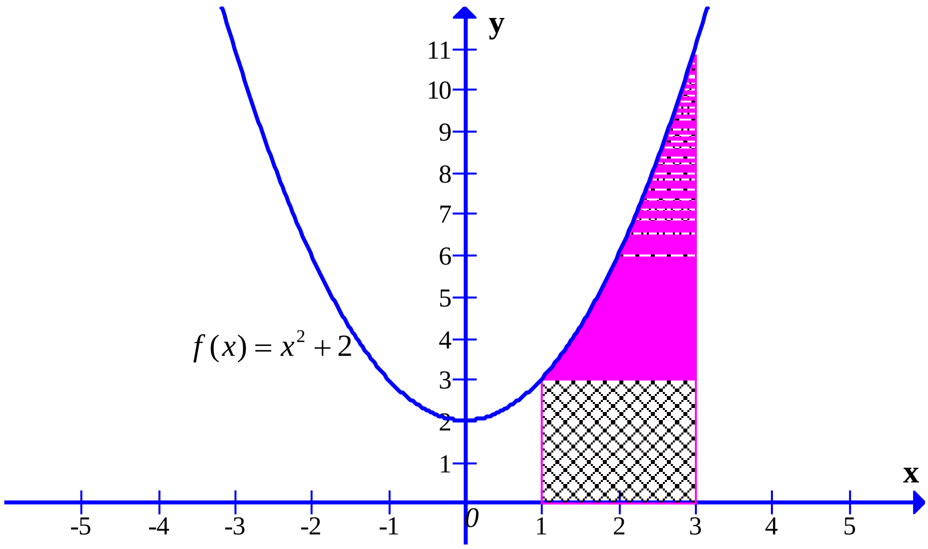
5.4


y
x
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
1
5
6
7
8
9
$f(x) = x^2 - 6x + 5$

5.5


y
x
$y = x^3 - 6x^2 + 11x - 6$
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-1
1
2
3
4
5
6
7

5.6


y
x
$f(x) = x^2 + 2$
11
10
9
8
7
6
5
4
3
2
1
0
-5
-4
-3
-2
-1
1
2
3
4
5

Chapter Summary

Notation

- The symbol for the integral is $\int$
- After the symbol for the integral we write the function we want to find the integral of, called the **integrand**
- Finally we write dx
- $\int 5x\ dx$

Integral symbol   Function to be integrated   with respect to x

The Indefinite Integral

- The indefinite integral is the expression $\int f(x)\ dx$ read as: **the indefinite integral of** $f(x)$ **with respect to** x.

Power Rule for the Indefinite Integral

$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C,\ \mathrm{n} \neq \text{-}1$	$\int x^7\ dx = \frac{x^{7+1}}{7+1} + C$ $= \frac{x^8}{8} + C$
$\int x^{-1}\ dx = \int \frac{dx}{x} = \ln\lvert x\rvert + C$	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$

The Sum and Difference Rules

$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$	$\int [3x+5]\ dx = \int 3x\ dx + \int 5\ dx$ $= \frac{3x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1} + C$ $= \frac{3}{2}x^2 + 5x + C$

Constant Multiple Rule

$\int k\, f(x)\ dx = k\int f(x)\ dx$	$\int 5x\ dx = 5\int x\ dx$ $= 5\frac{x^{1+1}}{1+1} + C$ $= \frac{5}{2}x^2 + C$

Rule Involving Functions in Exponential Form

$\int a^x \ dx = \frac{a^x}{\ln a} + C,\ a > 0$ and $a \neq 1$	$\int 3^x \ dx = \frac{3^x}{\ln 3} + C$

The Definite Integral

- A definite integral has a start and an end value.
- $\int_a^b f(x) \ dx$ reads: **the integral of** $f(x)$ **from** a to b.
- The numbers a and b are called the limits of integration, where $a < b$.

$\int_a^b f(x) \ dx = F(b) - F(a),\ a < b$	$\int_1^2 7x$ The indefinite integral is $7\int x \ dx = 7\left(\frac{x^2}{2}\right) + C$ $= \frac{7x^2}{2} + C$ at $x = 1$: $\frac{7(1)^2}{2} + C = \frac{7}{2} + C$ $x = 2: \frac{7(2)^2}{2} + C = \frac{7(4)}{2} + C$ $= 14 + C$ $\int_1^2 7x = (14 + C) - \left(\frac{7}{2} + C\right)$ $= 14 + C - \frac{7}{2} - C$ $= 14 - \frac{7}{2}$ $= \frac{28 - 7}{2}$ $= \frac{21}{2}$

Definite Integral Rules

- $\int_a^b kf(x) dx = k\int_a^b f(x) dx,\ k \in R$
- $\int_a^b [f(x) \pm g(x)] \ dx = \int_a^b f(x) dx \pm \int_a^b g(x) \ dx$

Area under a curve – region bounded by the given function, vertical lines and the x-axis

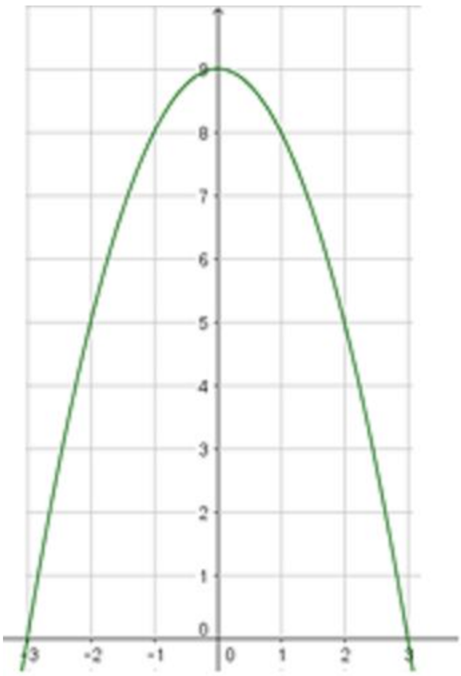
$$Area = \int_a^b f(x)\,dx$$

When calculating area under a curve $f(x)$ do the following:

- Sketch the area
- Determine the boundaries/limits a and b
- Set up the definite integral
- Integrate

Example

Find the area bounded by $f(x) = -x^2 + 9$ and the x-axis.



$-x^2 + 9 = 0$

$x^2 - 9 = 0$

$(x-3)(x+3) = 0$

$x - 3 = 0$ or $x + 3 = 0$

$\therefore x = 3$ or -3

The boundaries are $a = -3$ and $b = 3$

$$\int_{-3}^{3} (-x^2 + 9)\,dx = \left(-\frac{x^{2+1}}{2+1} + \frac{9x^{0+1}}{0+1} \right)_{-3}^{3}$$

$$= \left(-\frac{x^3}{3} + 9x \right)_{-3}^{3}$$

$$= \left(-\frac{3^3}{3} + 9(3) \right) - \left(-\frac{(-3)^3}{3} + 9(-3) \right)$$

$$= \left(-\frac{27}{3} + 27 \right) - \left(-\frac{-27}{3} - 27 \right)$$

$$= (-9 + 27) - (9 - 27)$$

$$= 18 - (-18)$$

$$= 18 + 18$$

$$= 36 \text{ square units}$$

Revision Exercise

Find each integral.

1. $\int (\frac{1}{2}x)dx$

2. $\int (-2x + 3)dx$

3. $\int (4x^2 + 3x - 1)dx$

4. $\int (8x^7 + 3x)dx$

5. Evaluate each definite integral.

5.1 $\int_0^1 (3x^2 + 7x + 1)dx$

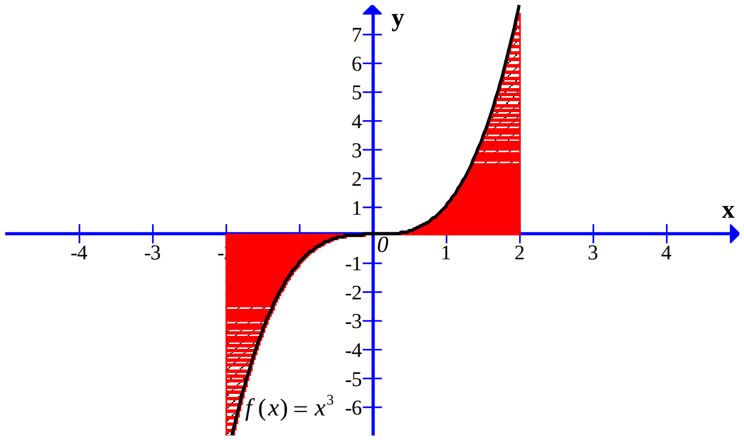
5.2 $\int_{-1}^0 (1 - x^2)dx$

5.3 $\int_4^5 (-7 + 6x + x^2)dx$

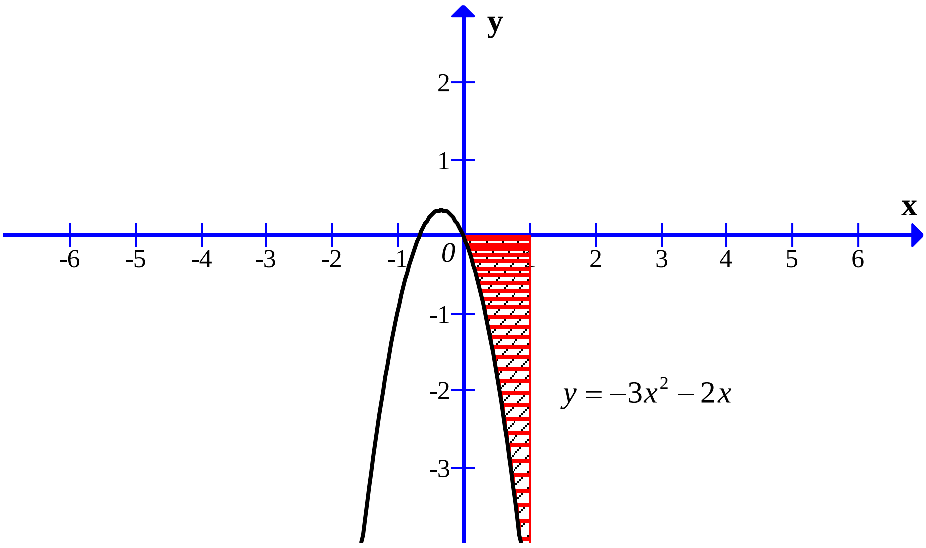
5.4 $\int_{-2}^3 (x + 2)(x - 3)dx$

6. Use integration to find the area of the shaded region.

6.1



6.2



7. Graph each of the following functions. Then, find the area between the function and the x-axis for the given interval using integration.

7.1 $f(x) = -2x + 3$ for $x = 1$ to $x = 4$

7.2 $f(x) = -x^2$ for $x = 0$ to $x = 5$

7.3 $f(x) = 9 - 3x^2$ for $x = 0$ to $x = 3$

7.4 $f(x) = x^3 - 4x$ for $x = -2$ to $x = 2$

7.5 $f(x) = -x^3$ for $x = -4$ to $x = 0$